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D-Branes and Scheme Theory

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In this highly speculative note we conjecture that it may be possible to understand some features of coincident D-branes, such as the appearance of enhanced non-abelian gauge symmetry, in a purely geometric fashion, using a form of geometry known as scheme theory. We give a very brief introduction to some relevant ideas from scheme theory, and point out how these ideas work in special cases.

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1 Introduction

As is well-known, on N coincident D-branes, $U(1)$ gauge symmetries are enhanced to $U(N)$ gauge symmetries, and scalars that formerly described normal motions of the branes become $U(N)$ adjoints. People have often asked what the deep reason for this behavior is – what does this tell us about the geometry seen by D-branes?

One possible answer is that these are indications that D-branes sense some sort of non-commutative geometry. However, as this enhancement to nonabelian gauge symmetry occurs even when $B \equiv 0$, if this is a sign of noncommutative geometry, it can not quite be noncommutative geometry in the same sense as [1, 2].

In this very short note, we shall speculate that this strange behavior may, at least in special cases, be an indication that the geometry seen by D-branes should be understood as a type of “information-preserving” geometry known as scheme theory.

Specifically, we begin by giving a very short introduction to some relevant ideas from scheme theory, including the idea that coincident subvarieties can be understood as “non-reduced schemes.” We then point out in a simple example (originally worked out in [3], for very different reasons) that branches of the classical moduli space of vacua can be understood in terms of moduli spaces of rank 1 sheaves on non-reduced schemes, and that $U(N)$ -adjoint-valued scalars also have a natural understanding.

Although this may sound somewhat new, in truth part of this is implicit in the idea that D-branes can be described, at some level, in terms of coherent sheaves [4]. Certainly coherent sheaves can be used to describe non-reduced subschemes of a variety, which we shall tentatively identify (in special cases) with coincident D-branes. What is new in this paper is that scheme theory may tie into D-branes at a much deeper level, by giving a purely geometric means for studying branches of the classical moduli space of vacua on a D-brane worldvolume, and by giving a mechanism for understanding how scalars describing normal motion can become $U(N)$ adjoints.

We do not claim to describe any particularly new or deep results, mathematical or physical, in this short note; rather, we shall only give a few easy calculations and review relevant calculations done elsewhere. This note does little more than point out some interesting ideas. We hope to publish more on this topic in the future.

The recent preprint [5] contains closely related ideas to those we shall present here. We worked on these ideas about two years ago, but decided to put off trying to write up any part of it until after a number of thorny technical issues had been resolved. However, the publication of [5] has motivated us to rapidly publish this very short overview of our ideas.

We should also pause to mention important differences between this paper and [5]. In [5], deformation quantization was used to generate recognizable D-brane features. By con-

trast, deformation quantization never enters this paper. Instead, we outline how essentially the same features of D-branes are already present in scheme theory, without resorting to deformation quantization, albeit in a rather subtle fashion.

A very different approach to understanding the non-abelian structure of coincident D-branes was also recently published in [6].

2 Information-preserving geometry

Scheme theory provides a notion of what we shall call an “information-preserving geometry.” What does that expression mean?

Consider a collision of two points. When two points collide, one is left with a single point. One does not know if a single point is really some number of coincident points, or if it is a single point. Thus, when thinking about motions of points (and higher dimensional subvarieties), the usual notions of geometry lose information – in the example above, there is no natural way to tell within the geometry if a single point represents a collision of multiple points. In order to distinguish the two possibilities, one would need to add additional¹ information.

By contrast, in scheme theory such information is automatically encoded in the geometry. In scheme theory not all points are identical – whether a point represents a collision of several other points is automatically encoded within the geometry itself. Thus, scheme theory is an example of an “information-preserving” geometry, and is a natural candidate for a purely geometrical means of distinguishing single D-branes from multiple coincident D-branes.

3 The skinny on fat points

How does scheme theory encode such additional information? The general idea is to work with the ring of functions² on a space, rather than the space itself. By knowing the functions, one can learn about the space.

For example, instead of talking about the space \mathbf{C}^2 , one works with the ring of holomorphic functions $\mathbf{C}[x, y]$. A single point at the origin in \mathbf{C}^2 is described by the ring $\mathbf{C}[x, y]/(x, y)$, where (x, y) denotes the ideal generated by x and y . Coincident points can

¹The reader might well object that this distinction is rather trivial – one could merely keep track of multiplicities of points. However, there is additional information that scheme theory also encodes, such as the relative tangent directions of collisions. We shall – briefly – return to this point later.

² In algebraic geometry, algebraic functions.

be described in a number of different ways, such as $\mathbf{C}[x, y]/(x^2, y)$ or $\mathbf{C}[x, y]/(x^2, xy, y^2)$ (depending upon the number of coincident points, among other things). The rings describing coincident points differ from the ring describing a single point, so we see explicitly how scheme theory distinguishes coincident points from single, ordinary points. (Such points are known as “fat points,” because they carry more information than a single point ordinarily would.) In fact, there is additional information here. The multiple ways of describing two coincident points encodes the relative directions from which they collided – so scheme theory can not only tell whether a single topological point is secretly the collision of several points, but also contains information about the relative directions from which they came.

A full introduction to scheme theory is far beyond the intended scope of this paper. For the interested reader, the standard basic reference is [7]; significantly more readable accounts, at varying levels of difficulty, include [8, 9, 10, 11]. In the interests of brevity, our discussion will rapidly become significantly more technical; however the interested reader should hopefully still be able to follow our presentation by frequently consulting the aforementioned references.

A scheme describing multiple coincident varieties is an example of a “non-reduced scheme,” and is an obvious candidate for a geometric description of coincident D-branes. This is precisely the conjecture we shall pursue – that one can understand many classical features of coincident D-branes in terms of non-reduced schemes. (Non-reduced schemes are more general than merely describing coincident subvarieties, but we shall not need the more general case here.)

As we are interested in how the $U(1)$ gauge symmetry on the worldvolume of a single D-brane becomes enhanced to a $U(N)$ gauge symmetry on N coincident D-branes, one of the first things we should study is sheaf theory on non-reduced schemes.

With that question in mind, what sorts of sheaves can appear on a non-reduced scheme? To be specific, let A be a ring and I a prime ideal in A . Define a non-reduced scheme $R = \text{Spec } A/I^n$, and the corresponding reduced scheme $C = \text{Spec } A/I$. In this description, R describes n coincident copies of C .

First, any coherent sheaf on C defines a coherent sheaf on R also. Specifically, it is straightforward to check that any (A/I) -module M is also an (A/I^n) -module, hence any coherent sheaf on C also defines a coherent sheaf on R . However, non-reduced schemes are subtle, and many statements that one would now be tempted to make are false. For example, although a bundle on C defines a coherent sheaf on R , strictly speaking a bundle on C does not define a bundle on R . For example, a rank 1 bundle on C is described by the module A/I , but a rank 1 bundle on R would be described by the module A/I^n , which is distinct.

Another set of examples of coherent sheaves on R are furnished by ideal sheaves. For example, take $A = k[x, y]$ for some field k , $I = (x) \subset A$, and define the ideal $J = (x, y)$ (i.e., the ideal generated by x and y). The module $J/I^2 \subset A/I^2$ defines an ideal sheaf on R . It is

interesting to note that the restriction of such an ideal sheaf to C generically has rank 1, but also has a torsion component. Specifically, the restriction of this ideal sheaf to C is defined by the module

$$(J/I^2) \otimes_{(A/I^2)} (A/I)$$

This module has two generators, namely $(x + I^2, 1 + I)$ and $(y + I^2, 1 + I)$, however the generator $(x + I^2, 1 + I)$ is annihilated by all elements of A/I other than the identity. Thus, this module defines a sheaf on C of the form $\mathcal{L} \oplus \mathcal{T}$, where \mathcal{L} is a locally free rank 1 sheaf (corresponding to generator $(y + I^2, 1 + I)$) and \mathcal{T} is a torsion sheaf with support at $x = y = 0$ (corresponding to generator $(x + I^2, 1 + I)$).

Now, it turns out that if R corresponds to r coincident copies of C , then rank r bundles on C often appear in the same families as rank 1 sheaves on R . (Note that for different sheaves to appear in the same flat family means they have the same Hilbert polynomial (with respect to some fixed projective embedding), and so appear on the same Hilbert scheme.) For example³, suppose $A = k[x, y]$ and $I = (x)$, as above. Define a one-parameter family of (A/I^2) -modules M_t as follows. Let M be a freely-generated (A/I) -module with two generators, say u and v . Construct a family M_t ($t \in k$) of (A/I^2) -modules by taking the underlying abelian group to be the same as that for M , leaving the y action on M invariant, and changing x to act on the generators u, v as

$$\begin{aligned} x : u &\longrightarrow tv \\ x : v &\longrightarrow 0 \end{aligned}$$

When $t = 0$, we see that x annihilates both generators, and in fact $M_0 \cong (A/I) \oplus (A/I)$. When $t \neq 0$, $M_t \cong A/I^2$.

Now that we have set up some basic technology, we shall describe a simple example.

4 A simple example

In [12] it was claimed that r D-branes wrapped on a curve C in a K3 of genus g_C were equivalent to a single D-brane wrapped on a curve of genus $g_R = r^2(g_C - 1) + 1$, lying in the linear system $|rC|$. Upon deeper reflection, however, the reader may wonder about this. If the twisted bundle endomorphism ϕ is nonzero, then this statement sounds reasonable – the eigenvalues of ϕ over any point of C define a cover of C inside the total space of the canonical bundle on C , and it seems reasonable to think of such ϕ as implying that the D-brane is wrapped on a cover of C .

But what about the degenerate limit in which the cover collapses down to C itself? In

³We would like to thank R. Donagi for pointing out this example to us.

this limit, the cover becomes a non-reduced scheme R , of arithmetic genus $g_R = r^2(g_C - 1) + 1$ (in agreement with the cover). This situation was discussed in [3], in the special case $r = 2$.

We claimed in the introduction that, just as a single D-brane has a $U(1)$ gauge field, features of r coincident D-branes can be recovered from Hilbert schemes of rank 1 sheaves on a corresponding non-reduced scheme. In [3, section 3], the Hilbert scheme of rank 1 sheaves on a non-reduced scheme $R = 2C$ is discussed. This Hilbert scheme has several disjoint components, whose closures intersect. One component consists of rank 2 bundles⁴ on C , but in addition, there are other components as well.

The physical interpretation of the first component is clear – we were expecting to find rank 2 bundles, after all. But what is the physical interpretation of the other components?

In order to understand the other components, take a moment to consider the classical moduli space of a two-dimensional gauge-theory with a single scalar ϕ transforming in the adjoint of $U(2)$. In particular, suppose ϕ is nilpotent, e.g.,

$$\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Such a ϕ is not gauge-equivalent to the zero matrix, yet all gauge-invariant observables, such as $\text{Tr } \phi$, $\text{Tr } \phi^2$, and $\det \phi$ vanish. So, if we want to study the classical moduli space of such a gauge theory, we need to be careful to find all possible Higgs branches. In particular, the “extra” components of a Hilbert scheme of rank 1 sheaves on $R = 2C$ correspond precisely to such nilpotent Higgs branches.

So far we have described components of Hilbert schemes of rank 1 sheaves on a non-reduced scheme $R = 2C$, and have given intuitive explanations of what these components should correspond to morally. However, a stronger statement can be made.

In [3, section 1], a precise description is given for how to explicitly deform Hilbert schemes of rank 1 sheaves on a non-reduced scheme $R = 2C$ to solutions of Hitchin’s equations for which the twisted-bundle-endomorphism ϕ is nilpotent (the “nilpotent cone”). In effect, [3, section 1] describes how to recover scalars that look like $N \times N$ matrices from non-reduced schemes. (Admittedly, however, such a description is implicit, not explicit, in their work.) In particular, our “intuitive explanations” for the components of the Hilbert scheme of rank 1 sheaves on R have a firm footing.

⁴The reader may be surprised to hear that a Hilbert scheme of rank 1 sheaves can include rank 2 sheaves, but recall that we worked out an example of this phenomenon at the end of the last section.

5 Nonlocality

One of the oft-quoted reasons for the popularity of noncommutative geometry is a certain notion of nonlocality. There is also a notion of nonlocality in scheme theory.

More precisely, by “nonlocality” we are referring to the fact that in scheme theory, there is typically no notion of a “small” open set – all open sets are very large. The essential difficulty is that closed sets are specified as vanishing loci of holomorphic functions – so closed sets are complex codimension one or higher. Therefore, all open sets see most of the space.

Put another way, the topologies that are most natural here are not the same topologies that one typically learns about in the context of differential geometry, for example.

This fact leads to a number of nontrivial technical complications when working in scheme theory. For example, since there are no “small” open sets, there is no such thing as a tubular neighborhood. Workarounds exist for these difficulties – instead of trying to use tubular neighborhoods to study locally how subvarieties sit inside an ambient space, one instead uses deformation arguments (in this case, deformation to the normal cone).

We do not have any strong statements to make about this notion of nonlocality, its physical interpretation in general, or its relationship to the nonlocality bandied about in noncommutative geometry specifically. We are merely pointing that ideas of nonlocality, at least loosely analogous to those in noncommutative geometry, also exist in scheme theory.

6 Too much information?

One significant problem with any attempt to describe coincident D-branes in terms of non-reduced schemes is that non-reduced schemes appear to encode more information than D-branes do physically. More precisely, when subvarieties of some space collide, the resulting non-reduced scheme not only encodes the number of coincident subvarieties, but also the relative directions from which they came. (For example, it is this extra information that distinguishes a Hilbert scheme of N points from a symmetric product.) Unfortunately, it is difficult to see how this extra information could appear physically.

The answer to this puzzle might be that scheme theory is only relevant for certain B field backgrounds, analogous (perhaps even equivalent) to the occurrence of noncommutative geometry [1, 2].

As a simple example, it was pointed out in [13] that this precise difficulty is solved for fat points by turning on a B field. More precisely, in [13] it was argued that in the

presence of a constant B field background, torsion-free sheaves on \mathbf{C}^2 appear physically (as “noncommutative $U(1)$ instantons”). (The B field appearing in [13] was of type $(1,1)$, and so only generated noncommutativity between holomorphic and nonholomorphic sectors. As emphasized to us by N. Nekrasov, if one restricts to holomorphic coordinates, the resulting ring is commutative, and so the noncommutative $U(1)$ instantons are truly the same thing as torsion-free sheaves.) Now, torsion-free sheaves encode the same information as fat points – the information about relative collision directions that confused us two paragraphs above appears physically as moduli of coincident noncommutative $U(1)$ instantons.

From the discussion in [13], we are led to suspect that scheme theory will only be physically relevant for certain nonzero B field backgrounds, as such backgrounds naively seem to enable us to encode the information naturally present in scheme theory.

7 Conclusions and bolder conjectures

In this short note we have conjectured that scheme theory may be relevant for studying D-branes, and given some limited evidence for this proposal.

If indeed scheme theory is relevant for describing D-branes classically, one is naturally led to several further conjectures:

1. There is, at least naively, a natural candidate within scheme theory to understand large N limits. Namely, it seems very natural to try to describe large N limits in terms of completions, or formal schemes [7, section II.9]. (This idea was also pointed out in [5].)
2. One might be able to use scheme theory to shed new light on wrapped D-branes. For example, one can build schemes that are morally K3 surfaces (in that their dualizing sheaves are trivial and H^1 of their structure sheaf vanishes), but whose underlying topological space is merely some rational normal scroll [14, 15]. (Such schemes are known in the literature as K3 carpets.) Perhaps this is some scheme-theoretic way to think about the topological twisting on wrapped D-branes pointed out in [12].
3. It might be possible to explicitly see small instantons splitting off and becoming D-branes (described as torsion sheaves), as different components of some Hilbert scheme.
4. If the nonabelian gauge symmetry on D-branes has a scheme-theoretic understanding, then perhaps the $\text{Spin}(32)/\mathbf{Z}_2$ gauge symmetry of type I strings also has a scheme-theoretic understanding, from viewing type I strings as D9-branes. It is difficult to see how such a gauge symmetry could arise in complex algebraic geometry; perhaps this arises from analogous ideas in real algebraic geometry.

5. One of the reasons mathematicians are fond of scheme theory is that it gives a unified framework in which to study both algebraic geometry and number theory. Perhaps some sort of scheme-theoretic approach to D-branes could be used to shed light on the number-theoretic ideas in [16, 17].

We would again like to emphasize that we do not presently feel the ideas presented in this short note to be more than enticing conjectures. We hope to report on further developments along these lines at some point in the future.

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